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The singular sources method for an inverse problem with mixed boundary conditions

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Abstract

We use the singular sources method to detect the shape of the obstacle in a mixed boundary value problem. The basic idea of the method is based on the singular behavior of the scattered field of the incident point-sources on the boundary of the obstacle. Moreover we take advantage of the scattered field estimate by the backprojection operator. Also we give a uniqueness proof for the shape reconstruction.

Keywords: Inverse scattering; Mixed boundary conditions; Singular sources method

1. Introduction

The problem of the reconstruction of an obstacle from some knowledge of the scattered wave at large distance is a well-known problem in the area of the inverse problems. There are several methods for the shape reconstruction in the literature [1,4,9]. One of the important property of a reconstruction method in the applications is its independence of the knowledge of the boundary condition. In fact from physical point of view, it is not realistic to know the boundary condition. The linear sampling method and singular sources method

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have this property. In [9] these methods have been applied for sound-soft and hard-soft obstacles, when the boundary condition is Dirichlet or Neumann.

Recently a kind of obstacle scattering problems has been introduced in electromagnetic scattering which is called partially coated obstacle scattering. These obstacles are coated by some material on a portion of the boundary to reduce the radar cross section of the scattered wave. Here the boundary condition of the obstacle is a kind of mixed boundary condition [2,3,6]. A similar kind of the inverse problem of the mixed boundary conditions appears in acoustic scattering.

The authors in [1] proposed the linear sampling method to reconstruct the shape of the obstacle in the case of mixed boundary conditions. The same authors have used a variational method for determining the essential supremum of the surface impedance in [2]. Also in [5] the point-source method has been applied to reconstruct the coated portion and the surface impedance under the assumption that the shape is determined.

In the present paper we use the singular sources method to reconstruct the shape of the obstacle in a mixed boundary conditions model. This method is proposed by Potthast in [10]. In this method the reciprocity relation is used to derive the backprojection operator. This operator enable us to estimate the scattered field from the far field pattern. The basic idea of the method is based on the singularity behavior of the scattered field of point-source on the boundary of the obstacle. This means that if $\Phi^s(\,.\,,z)$ denote this scattered field and z tends to the boundary then

$$\left|\Phi^s(z,z)\right| \to \infty.$$

This behavior shows that the boundary is the set of points where the scattered field $\Phi^s(z,z)$ becomes singular. In order to apply the singular sources method we need to estimate $\Phi^s(z,z)$ from the knowledge of the far field pattern, $u^\infty(\hat{x},d)$, which is derived by the backprojection operator.

In Section 2, the direct scattering problem is considered. In this section also we prove a lemma which estimates the norm of the point-source on the obstacle, i.e. $\|\Phi(\,.\,,z)\|_{H^1(D)}$. In Section 3, by using this lemma we show the singular behavior of the scattered field $\Phi^s(z,z)$. We will prove the uniqueness of the shape reconstruction of the partially coated scattering in this section, too. Finally in Section 3, we will apply the singular sources method to reconstruct the shape of the obstacle.

2. Partially coated scattering

In this section we formulate the direct scattering problem for mixed boundary value problem. Then we find an upper bound for the point source which is crucial in the next section.

Let $D \subseteq \mathbb{R}^m$ $(m = 2, 3)$ be an open bounded domain with Lipschitz boundary, Γ such that $\mathbb{R}^m \setminus \bar{D}$ is connected. We also assume that the boundary, Γ has a Lipschitz dissection $\Gamma = \Gamma_D \cup \Pi \cup \Gamma_I$, where Γ_D and Γ_I are C^2, disjoint, relatively open subsets of Γ, and Π is their common boundary in Γ.

We consider a plane wave $u^i(x,d) = e^{ikx.d}$, in the direction of d with $|d| = 1$. Let $u^s(x,d)$ be the solution of the following exterior mixed boundary value problem:

$$\begin{aligned} &\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^m \setminus \bar{D}, \\ &u = f \quad \text{on } \Gamma_D, \\ &\frac{\partial u}{\partial \nu} + i\lambda(x) u = h \quad \text{on } \Gamma_I, \end{aligned} \tag{1}$$

for $f = -u^i|_{\Gamma_D}$ and $h = -\frac{\partial u^i}{\partial \nu} - i\lambda u^i|_{\Gamma_I}$. Here $k > 0$ is the wave number, ν denotes the unit outward normal vector which is defined on $\Gamma_D \cup \Gamma_I$. The character λ is a real non-negative function and $\lambda \in L^\infty(\Gamma_I)$. Moreover, the scattered wave u^s is required to satisfy Sommerfeld radiation condition

$$\lim_{|x| \to \infty} |x|^{(m-1)/2} \left(\frac{\partial u^s}{\partial \nu} - iku^s \right) = 0, \tag{2}$$

uniformly in all directions $\hat{x} := x/|x|$.

The radiation condition (2) implies an asymptotic behavior of the function,

$$u^s(x, d) = \frac{e^{ik|x|}}{|x|^{(m-1)/2}} u^\infty(\hat{x}, d) + o\big(|x|^{-(m+1)/2}\big), \tag{3}$$

uniformly in all direction $\hat{x} = x/|x|$, see [4]. The amplitude factor u^∞ is known as the far field pattern of the scattered wave, u^s. Notice that u^∞ is a function of the incident direction $d \in \Omega$, and the observation direction $\hat{x} \in \Omega$. The fundamental solution of Helmholtz equation is given by

$$\Phi(x, y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x - y|), & x \neq y,\ m = 2, \\ \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, & x \neq y,\ m = 3, \end{cases}$$

where $H_0^{(1)}$ denotes the Hankel function of order zero and of the first kind [4].

In order to investigate the problem (1) we need to recall the definition of the following Sobolev spaces. Let $\Gamma_0 \subseteq \Gamma$ be a portion of the boundary, Γ. If $H^1(D)$ denotes the usual Sobolev space and $H^{1/2}(\Gamma)$ its usual trace space, then we define

$$\begin{aligned} &H^{1/2}(\Gamma_0) := \big\{ u|_{\Gamma_0} \colon u \in H^{1/2}(\Gamma) \big\}, \\ &\tilde{H}^{1/2}(\Gamma_0) := \big\{ u \in H^{1/2}(\Gamma) \colon \operatorname{supp} u \subseteq \bar{\Gamma}_0 \big\}, \\ &H^{-1/2}(\Gamma_0) := \big(\tilde{H}^{1/2}(\Gamma_0) \big)^* \quad \text{the dual space of } \tilde{H}^{1/2}(\Gamma_0), \\ &\tilde{H}^{-1/2}(\Gamma_0) := \big(H^{1/2}(\Gamma_0) \big)^* \quad \text{the dual space of } H^{1/2}(\Gamma_0). \end{aligned}$$

In [3], it is shown that for every $f \in H^{1/2}(\Gamma_D)$ and $g \in H^{-1/2}(\Gamma_I)$, the exterior mixed boundary value problem (1) under the condition (2) has a unique weak solution, for constant λ. Furthermore, in [3], it is shown that the solution is in $H^1_{\mathrm{loc}}(\mathbb{R}^m \setminus \bar{D})$. Although in [3], it is assumed that λ is constant, but all of the above results remain valid if $\lambda \in L^\infty(\Gamma_I)$ and $\lambda \geqslant 0$, as Colton and Cakoni have been indicated in [2].

Suppose that $\Phi^s(\,.\,, z)$, $z \in \mathbb{R}^m \setminus \bar{D}$, is the scattered wave of the incident wave $\Phi(\,.\,, z)$. This means that $\Phi^s(\,.\,, z)$ is the solution of (1) and (2) with the boundary condition $f = -\Phi(\,.\,, z)|_{\Gamma_D}$ and $h = -\frac{\partial \Phi(\,.\,, z)}{\partial \nu} - i\lambda \Phi(\,.\,, z)|_{\Gamma_I}$. We also denote the far field pattern of $\Phi^s(\,.\,, z)$ by $\Phi^\infty(\hat{x}, z)$. We extend λ to the whole of the boundary with the definition zero on Γ_D. Now we note that for every $z \in \mathbb{R}^m \setminus \bar{D}$, there are $\psi_I(\,.\,,z) \in \tilde{H}^{1/2}(\Gamma_I)$ and $\psi_D(\,.\,,z) \in \tilde{H}^{-1/2}(\Gamma_D)$ such that the following relations on ∂D are satisfied:

$$\Phi^s(\,.\,,z)|_\Gamma = -\Phi(\,.\,,z)|_\Gamma + \psi_I(\,.\,,z),$$
$$\left(\frac{\partial \Phi^s(\,.\,,z)}{\partial \nu} + i\lambda \Phi^s(\,.\,,z)\right)\bigg|_\Gamma = \left(-\frac{\partial \Phi(\,.\,,z)}{\partial \nu} - i\lambda \Phi(\,.\,,z)\right)\bigg|_\Gamma + \psi_D(\,.\,,z).$$

By consider the continuous dependence of the scattered field on the boundary data which is proved in [3], by formula (22), we obtain

$$\begin{aligned}&\|\psi_D(\,.\,,z)\|_{\tilde{H}^{-1/2}(\Gamma_D)}, \|\psi_I(\,.\,,z)\|_{\tilde{H}^{1/2}(\Gamma_I)}\\ &\quad \leqslant C\left(\|\Phi(\,.\,,z)\|_{H^{1/2}(\Gamma)} + \left\|\frac{\partial \Phi}{\partial \nu}(\,.\,,z)\right\|_{H^{-1/2}(\Gamma)}\right).\end{aligned}$$

Now, from the above relation, Theorem 3.37 and Lemma 4.3 in [8], we conclude

$$\|\Phi(\,.\,,z)\|_{H^{1/2}(\Gamma)}, \left\|\frac{\partial \Phi}{\partial \nu}(\,.\,,z)\right\|_{H^{-1/2}(\Gamma)} \leqslant C\|\Phi(\,.\,,z)\|_{H^1(D)}, \tag{4}$$

for every $z \in \mathbb{R}^m \setminus \bar{D}$. By the following proposition, we estimate the functions ψ_D and ψ_I.

Proposition 1. (i) *If $D \subset \mathbb{R}^2$, then there exist constants $\tau, c > 0$, such that*

$$\|\psi_D(\,.\,,z)\|^2_{\tilde{H}^{-1/2}(\Gamma_D)}, \|\psi_I(\,.\,,z)\|^2_{\tilde{H}^{1/2}(\Gamma_I)} \leqslant c\big|\ln d(z,D)\big|,$$

for every $z \notin D$, which satisfy $0 < d(z,D) < \tau$. Moreover for every $z \in \mathbb{R}^2 \setminus \bar{D}$, we have

$$\|\psi_D(\,.\,,z)\|^2_{\tilde{H}^{-1/2}(\Gamma_D)}, \|\psi_I(\,.\,,z)\|^2_{\tilde{H}^{1/2}(\Gamma_I)} \leqslant C\big|\ln d(z,D)\big| + E,$$

where the constants C and E depend only on D.

(ii) *If $D \subset \mathbb{R}^3$, then for every $z \in \mathbb{R}^3 \setminus \bar{D}$, we have*

$$\|\psi_D(\,.\,,z)\|^2_{\tilde{H}^{-1/2}(\Gamma_D)}, \|\psi_I(\,.\,,z)\|^2_{\tilde{H}^{1/2}(\Gamma_I)} \leqslant \frac{c}{d(z,D)}.$$

By considering the relation (4) and the following lemma the proof of the above proposition will be clear.

Lemma 2. (i) *If $D \subset \mathbb{R}^2$, then there exist constants $\tau, c > 0$, such that*

$$\|\Phi(\,.\,,z)\|^2_{H^1(D)} \leqslant c\big|\ln d(z,D)\big|,$$

for every $z \notin D$, which satisfy $0 < d(z,D) < \tau$. Moreover for every $z \in \mathbb{R}^2 \setminus \bar{D}$, we have

$$\|\Phi(\,.\,,z)\|^2_{H^1(D)} \leqslant C\big|\ln d(z,D)\big| + E,$$

where the constants C and E depend only on D.

(ii) *If $D \subset \mathbb{R}^3$, then for every $z \in \mathbb{R}^3 \setminus \bar{D}$, we have*

$$\|\Phi(\,.\,,z)\|^2_{H^1(D)} \leqslant \frac{c}{d(z,D)}.$$

Proof. (i) Suppose in (1) we have $k=0$, and $\Phi_0(x,z)$ is the fundamental solution, in this case. The relation (3.61) in [4] implies that $\Phi(x,z)-\Phi_0(x,z)$ is differentiable for all $x,z\in\mathbb{R}^2$, hence $\|\Phi(\,.\,,z)-\Phi_0(\,.\,,z)\|^2_{H^1(D)}$ is bounded. Thus it is sufficient to prove

$$\big\|\Phi_0(\,.\,,z)\big\|^2_{H^1(D)}\leqslant C\big|\ln d(z,D)\big|+E, \tag{5}$$

for every $z\in\mathbb{R}^2\setminus\bar{D}$. In order to show this we have

$$\Phi_0(x,z)=\frac{1}{2\pi}\ln\frac{1}{|x-z|},$$
$$\nabla_x\Phi_0(x,z)=-\frac{1}{2\pi}\frac{x-z}{|x-z|^2}.$$

Therefore

$$\begin{aligned}\big\|\Phi_0(\,.\,,z)\big\|^2_{H^1(D)}&\leqslant C_1\int_D\frac{1}{|x-z|^2}+\left(\ln\frac{1}{|x-z|}\right)^2dx\leqslant C_2\int_D\frac{1}{|x-z|^2}\,dx\\&=C_2\int_{D\cap B_R(z)}\frac{1}{|x-z|^2}\,dx+C_2\int_{D\setminus B_R(z)}\frac{1}{|x-z|^2}\,dx,\end{aligned}$$

where $B_R(z)$ is the ball with the center, z and radius R. The second integral is bounded because of the boundedness of D and $|x-z|>R$. Also if $d(z,D)=h$, then for every $x\in D\cap B_R(z)$, we have $h\leqslant|x-z|\leqslant R$, so the first integral is bounded from above by

$$C\int_h^R\frac{2\pi r\,dr}{r^2}\leqslant E\ln\frac{R}{h}.$$

Therefore, there are constants $C,E>0$, such that for every h,

$$\big\|\Phi_0(\,.\,,z)\big\|^2_{H^1(D)}\leqslant C+E\ln\frac{1}{h}.$$

This relation is the same as (5) which proves the desired inequalities for every z and also for small values of h.

(ii) Similar to the dimension two, we consider the fundamental solution, $\Phi_0(x,z)$ in the case $k=0$, and note that we have

$$\big|\Phi_0(x,z)\big|=\frac{1}{|x-z|},$$
$$\big|\nabla_x\Phi_0(x,z)\big|=\frac{1}{|x-z|^2}.$$

Therefore we can write

$$\begin{aligned}\big\|\Phi_0(\,.\,,z)\big\|^2_{H^1(D)}&=\int_D\left(\frac{1}{|x-z|^2}+\frac{1}{|x-z|^4}\right)dx\\&=\int_{D\cap B_R(z)}\left(\frac{1}{|x-z|^2}+\frac{1}{|x-z|^4}\right)dx\end{aligned}$$

$$+\int_{D\setminus B_R(z)}\left(\frac{1}{|x-z|^2}+\frac{1}{|x-z|^4}\right)dx$$
$$\leqslant\int_h^R C\pi r^2\left(\frac{1}{r^2}+\frac{1}{r^4}\right)dr+C\leqslant C\pi\left(R-h+\frac{1}{h}-\frac{1}{R}\right)+C.$$

Hence, there are constants $C, E>0$, such that for every h,

$$\begin{aligned}\|\Phi(\,.\,,z)\|^2_{H^1(D)}&\leqslant\|\Phi_0(\,.\,,z)\|^2_{H^1(D)}+\|\Phi(\,.\,,z)-\Phi_0(\,.\,,z)\|^2_{H^1(D)}\\&\leqslant C+E\left(\frac{1}{h}-h\right).\end{aligned}$$

Thus for $c=E+\frac{C^2}{4E}$ we have

$$\|\Phi(\,.\,,z)\|^2_{H^1(D)}\leqslant\frac{c}{d(z,D)}.\qquad\square$$

3. Shape reconstruction

The inverse obstacle scattering problem is to determine the shape of the obstacle and to recover the boundary conditions on the obstacle from the far field pattern, $u^\infty(\hat{x},d)$ for all directions $\hat{x}, d\in\Omega$. In the inverse mixed boundary value problem, the main goal is to determine Γ, Γ_D, Γ_I and λ from $u^\infty(\hat{x},d)$. In this section we reconstruct Γ by information of $u^\infty(\hat{x},d)$ in all incident directions $d\in\Omega$ and all observation directions $\hat{x}\in\Omega$. In [1] and [3], the linear sampling method is used to recover the boundary, Γ. Cakoni and Colton have used a variational method for determining the essential supremum of the surface impedance, λ, in [2]. Also Kress and Rundell have employed a Newton method to recover the shape, D, and impedance, λ, in [7], when obstacle is soft (i.e. $\Gamma_D=\emptyset$), and the boundary is starlike. Recently in [5] the point-source method is used to determine Γ_D, Γ_I and λ, by knowing the shape of D.

In this section we develop the singular sources method to reconstruct the shape of the scattering object. This method is used in [9] to reconstruct the shape of a scatterer without having the boundary condition or physical properties of the scatterer, in the cases of soft obstacle, hard obstacle and inhomogeneous medium scattering. In this method we use the field $\Phi^s(z,z)$ to reconstruct the shape of the scattering object. The boundary, Γ, is found as the set of points where $\Phi^s(z,z)$ becomes singular.

In the following theorem we now investigate the behavior of $\Phi^s(z,z)$ when z tends to the boundary. Here D is the same as before, moreover Γ_D and Γ_I are assumed to be C^2.

Theorem 3. *Let $\Phi^s(\,.\,,z)$ be the scattering field of the point-source $\Phi(\,.\,,z)$ by a mixed boundary condition scatterer D, moreover Γ_D and Γ_I are C^2. If z tends to a point of $\Gamma_D\cup\Gamma_I$, then*

$$|\Phi^s(z,z)|\to\infty.$$

Proof. We prove the theorem in the case of dimension 3. The proof for the case of dimension 2 is similar. Suppose that $z \to z^* \in \Gamma$, then we consider two cases.

Case 1. Let $z^* \in \Gamma_D$. We consider $\psi_I(\,.\,,z) \in H^{1/2}(\Gamma)$ as an extension of $\psi_I(\,.\,,z) \in \tilde{H}^{1/2}(\Gamma_I)$ by zero on the whole boundary, Γ. Now suppose $w_D(\,.\,,z)$ is the radiating solution of Helmholtz equation

$$\Delta w_D + k^2 w_D = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D},$$
$$w_D = \psi_I(\,.\,,z) \quad \text{on } \Gamma.$$

Thus $w_D(\,.\,,z) \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ and

$$\left\| \chi w_D(\,.\,,z)\right\|^2_{H^1(\mathbb{R}^3\setminus\bar{D})} \leqslant C \left\| \psi_I(\,.\,,z)\right\|^2_{H^{1/2}(\Gamma)},$$

where $\chi \in C_0^\infty(\mathbb{R}^m)$ is an arbitrary cut-off function. Then by Proposition 1, we conclude that for every z, we have

$$\left\| w_D(\,.\,,z)\right\|^2_{H^1(B\setminus\bar{D})} \leqslant \frac{C}{d(z,D)}, \tag{6}$$

where B is a ball contains D. Let $u_D(\,.\,,z)$ be the radiating solution of Helmholtz equation with the following boundary condition:

$$u_D(\,.\,,z) = -\Phi(\,.\,,z) \quad \text{on } \Gamma.$$

By uniqueness of the radiating solution from the boundary condition, we obtain

$$\Phi^s(\,.\,,z) = w_D(\,.\,,z) + u_D(\,.\,,z). \tag{7}$$

On the other hand Theorem 2.1.15 in [9] implies that for every z near D, u_D satisfies the following estimate:

$$\left| u_D(z,z)\right| \geqslant \frac{c}{d(z,D)}. \tag{8}$$

Now we show that for z near D, the rate of growth of $w_D(z,z)$ is less than the rate of growth of $\frac{1}{d(z,D)}$. That is

$$\left| w_D(z,z)\right| \leqslant \frac{c}{d(z,D)^{1/2}}.$$

In order to see this estimate, let G_1 and G_2 be two neighborhoods of z^*, with $\bar{G}_1 \Subset G_2$ and $G_2 \cap \Gamma \subseteq \Gamma_D$. Since support of ψ_I is located in Γ_I, so $\psi_I|_{\Gamma_D} = 0$. Then notice that Γ_D is smooth and $\psi_I \in H^{3/2}(G_2 \cap \Gamma)$, thus theorem of regularity of the solution up to the boundary in [8] implies that $w_D(\,.\,,z) \in H^2(\Omega_1)$, and

$$\left\| w_D(\,.\,,z)\right\|_{H^2(\Omega_1)} \leqslant C\left(\left\| w_D(\,.\,,z)\right\|_{H^1(\Omega_2)} + \|\psi_I\|_{H^{3/2}(G_2\cap\Gamma)}\right),$$

where $\Omega_i = G_i \setminus \bar{D}$. Hence by the relation (6), we have

$$\left\| w_D(\,.\,,z)\right\|^2_{H^2(\Omega_1)} \leqslant \frac{C}{d(z,D)}.$$

On the other hand the imbedding theorem implies that $w_D(\,.\,,z)$ is a Hölder continuous function and

$$\left| w_D(x,z)\right| \leqslant C \left\| w_D(\,.\,,z)\right\|_{H^2(\Omega_1)},$$

for every $x \in \Omega_1$. Therefore

$$\left|w_D(z,z)\right| \leqslant \frac{c}{d(z,D)^{1/2}}, \tag{9}$$

for every $z \in \Omega_1$. By considering (7)–(9) the proof of theorem is complete in this case.

Case 2. Let $z^* \in \Gamma_I$. Similar to the first case consider $\psi_D(\,.\,,z) \in H^{-1/2}(\Gamma)$ as an extension of $\psi_D(\,.\,,z) \in \tilde{H}^{-1/2}(\Gamma_D)$ by zero on the whole boundary. Now let $w_I(\,.\,,z)$ be the radiating solution of Helmholtz equation

$$\Delta w_I + k^2 w_I = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D},$$
$$\frac{\partial w_I}{\partial \nu} + i\lambda w_I = \psi_D(\,.\,,z) \quad \text{on } \Gamma.$$

Similarly, $w_I(\,.\,,z) \in H^1_{\text{loc}}(\mathbb{R}^3 \setminus \bar{D})$ and

$$\left\|w_I(\,.\,,z)\right\|^2_{H^1(B\setminus\bar{D})} \leqslant \frac{C}{d(z,D)}.$$

In this case

$$\Phi^s(\,.\,,z) = w_I(\,.\,,z) + u_I(\,.\,,z),$$

where $u_I(\,.\,,z)$ is the radiating solution of Helmholtz equation with the following boundary condition:

$$\frac{\partial u_I}{\partial \nu} + i\lambda u_I = -\frac{\partial \Phi(\,.\,,z)}{\partial \nu} - i\lambda \Phi(\,.\,,z) \quad \text{on } \Gamma.$$

We claim that u_I and w_I satisfy the following estimations:

$$\left|u_I(z,z)\right| \geqslant \frac{c}{d(z,D)},$$
$$\left|w_I(z,z)\right| \leqslant \frac{c}{d(z,D)^{1/2}}.$$

In order to see them, let v be the radiating solution of Helmholtz equation with the following boundary condition:

$$\frac{\partial v}{\partial \nu} = i\lambda \quad \text{on } \Gamma.$$

Then $w = e^v w_I$ satisfy

$$\Delta w + k^2 w = f \quad \text{in } \mathbb{R}^3 \setminus \bar{D},$$
$$\frac{\partial w}{\partial \nu} = e^v \psi_D(\,.\,,z) \quad \text{on } \Gamma,$$

where $f = e^v(|\nabla v|^2 w_I + 2\nabla v.\nabla w_I - k^2 v w_I)$. Now similar to the first case, by theorem of regularity of the solution up to the boundary in the neighborhood of z^*, we conclude that $w \in H^2(\Omega_1)$, and we have

$$\|w\|_{H^2(\Omega_1)} \leqslant C\big(\|w\|_{H^1(\Omega_2)} + \|e^v \psi_D\|_{H^{3/2}(G_2\cap\Gamma)} + \|f\|_{L^2(\Omega_2)}\big).$$

Also here we have $\psi_D|_{\Gamma_I} = 0$, and

$$
\begin{aligned}
&\|w\|_{H^1(\Omega_2)} \leqslant C\|w_I\|_{H^1(B\setminus\bar{D})} \leqslant \frac{C}{d(z,D)^{1/2}},\\
&\|f\|_{L^2(\Omega_2)} \leqslant C\|w_I\|_{H^1(B\setminus\bar{D})} \leqslant \frac{C}{d(z,D)^{1/2}},\\
&\partial_i\partial_j w = e^v(\partial_i\partial_j v + \partial_i v\partial_j v w_I + \partial_i v\partial_j w_I + \partial_j v\partial_i w_I + \partial_i\partial_j w_I).
\end{aligned}
$$

Now since v is bounded on Ω_2, $0 < m \leqslant |e^v| = e^{Re(v)}$, then

$$
\begin{aligned}
m\|\partial_i\partial_j w_I\|_{L^2(\Omega_1)} &\leqslant \|\partial_i\partial_j w\|_{L^2(\Omega_1)} + \|\partial_i\partial_j w - e^v\partial_i\partial_j w_I\|_{L^2(\Omega_1)}\\
&\leqslant \|w\|_{H^2(\Omega_1)} + C\|w_I\|_{H^1(\Omega_1)} \leqslant \frac{C}{d(z,D)^{1/2}}.
\end{aligned}
$$

Therefore $\|w_I\|_{H^2(\Omega_1)} \leqslant \frac{C}{d(z,D)^{1/2}}$, and by imbedding theorem we have

$$
\left|w_I(z,z)\right| \leqslant \frac{c}{d(z,D)^{1/2}},
$$

for every $z \in \Omega_1$. It is remained to show that

$$
\left|u_I(z,z)\right| \geqslant \frac{c}{d(z,D)},
$$

for every z near to the boundary, Γ. Similar to the above we change u_I to $u = e^v u_I$. Now we can write $u = u^1 + u^2$, where u^1 is the radiating solution of

$$
\begin{aligned}
&\Delta u^1 + k^2 u^1 = 0 \quad \text{in } \mathbb{R}^3\setminus\bar{D},\\
&\frac{\partial u^1}{\partial\nu} = -\frac{\partial(e^v\Phi(\,.\,,z))}{\partial\nu} \quad \text{on } \Gamma,
\end{aligned}
$$

and u^2 satisfies in the following equation:

$$
\begin{aligned}
&\Delta u^2 + k^2 u^2 = f \quad \text{in } \mathbb{R}^3\setminus\bar{D},\\
&\frac{\partial u^2}{\partial\nu} = 0 \quad \text{on } \Gamma.
\end{aligned}
$$

Similar to the above we can get

$$
\|u^2\|_{H^2(\Omega_1)} \leqslant C\|f\|_{L^2(\Omega_2)} \leqslant C\|u_I\|_{H^1(\Omega_2)} \leqslant \frac{C}{d(z,D)^{1/2}}.
$$

Thus

$$
\left|u^2(z,z)\right| \leqslant \frac{c}{d(z,D)^{1/2}}.
$$

By considering the boundedness of e^v in a neighborhood of D, the proof will be complete if we show that

$$
\left|u^1(z,z)\right| \geqslant \frac{c}{d(z,D)}.
$$

This estimate follows from the following lemma. □

Lemma 4. *Let $u(\,.\,,z)$ be the radiating solution of Helmholtz equation with the boundary condition*

$$\frac{\partial u}{\partial \nu} = -\frac{\partial(w\Phi(\,.\,,z))}{\partial \nu} \quad \text{on } \Gamma,$$

where w is a smooth function defined in the exterior of D. In the case of dimension two the following estimate holds for z near D:

$$\left|u(z,z)\right| \geqslant c\left|\ln d(z,D)\right|.$$

In the case of dimension three this estimate will be

$$\left|u(z,z)\right| \geqslant \frac{c}{d(z,D)}.$$

Proof. Theorem 2.1.15 in [9] establish similar results for the solution with the boundary condition

$$\frac{\partial u}{\partial \nu} = -\frac{\partial \Phi(\,.\,,z)}{\partial \nu} \quad \text{on } \Gamma,$$

and our result can be derived in the same way if we replace $\Phi(\,.\,,z)$ by $w\Phi(\,.\,,z)$. □

Before we begin to consider the shape reconstruction problem, we investigate the uniqueness of the reconstruction. This uniqueness result plays an important role in the shape reconstruction. The precise meaning of uniqueness in the mixed boundary value problem is that if from the knowledge of the far field we can reconstruct Γ, Γ_D, Γ_I and λ uniquely. This question has been answered exactly in [5]. In the following theorem we only show the uniqueness of the reconstruction of the boundary, Γ, with simpler proof which is used the singular behavior of $\Phi^s(z,z)$. In the proof of this theorem we will use the reciprocity relation respect to the case of mixed boundary conditions from [5].

Theorem 5 (Mixed reciprocity relation). *For the acoustic scattering of the plane waves $u^i(\,.\,,d)$, $d \in \Omega$, and the point sources $\Phi(\,.\,,z)$, $z \in \mathbb{R}^m \setminus \bar{D}$, from a mixed boundary condition scatterer D we have*

$$\Phi^\infty(\hat{x},z) = \gamma_m u^s(z,-\hat{x}), \quad z \in \mathbb{R}^m \setminus \bar{D},\ \hat{x} \in \Omega,$$

where

$$\gamma_m = \begin{cases} \frac{e^{i\pi/4}}{\sqrt{8\pi k}}, & m = 2, \\ \frac{1}{4\pi}, & m = 3. \end{cases}$$

Theorem 6. *Let D_1 and D_2 be mixed boundary condition obstacles. If the far field patterns $u_1^\infty(\hat{x},d)$ and $u_2^\infty(\hat{x},d)$ for both scatterers coincide for all $\hat{x}, d \in \Omega$, then $D_1 = D_2$.*

Proof. Let G be the unbounded component of the complement of $\bar{D}_1 \cup \bar{D}_2$. From

$$u_1^\infty(\hat{x},d) = u_2^\infty(\hat{x},d) \quad \text{for all } \hat{x}, d \in \Omega,$$

and Rellich lemma in [4], we obtain

$$u_1^s(x,d) = u_2^s(x,d) \quad \text{for all } x \in G,\ d \in \Omega.$$

Thus by Theorem 5 we have

$$\Phi_1^\infty(\hat{x}, z) = \Phi_2^\infty(\hat{x}, z) \quad \text{for all } \hat{x} \in \Omega,\ z \in G.$$

Again we can use Rellich lemma to achieve the following relation:

$$\Phi_1^s(x,z) = \Phi_2^s(x,z) \quad \text{for all } x, z \in G. \tag{10}$$

Suppose that $D_1 \neq D_2$. Now without loss of generality, we can assume that there exists $z_0 \in \partial G$ such that $z_0 \in \partial D_1 \setminus D_2$. Then by Theorem 3 we conclude that

$$\infty > \Phi_2^s(z_0, z_0) = \lim_{z \to z_0,\, z \in G} \Phi_2^s(z,z) = \lim_{z \to z_0,\, z \in G} \Phi_1^s(z,z) = \infty.$$

This contradiction shows that $D_1 = D_2$. $\square$

Now we apply the singular sources method to reconstruct the shape of the obstacle. According to Theorem 3, the boundary is the set of points where $\Phi^s(z,z)$ is large. In order to determine this set we should calculate $\Phi^s(z,z)$ from the far field pattern $u^\infty(\hat{x}, d)$.

Suppose we know a priori information $D \subset B$, where B is a bounded domain. For every $z \in B$, let $G(z)$ be a smooth region which does not have Dirichlet eigenvalue $-k^2$, $z \notin G(z)$ and $\bar{D} \subseteq G(z) \subseteq B$, where k is the wave number. For every ε there is $g \in L^2(\Omega)$ such that

$$\left\| \Phi(\,.\,,z) - v_g \right\|_{L^2(\partial G)} < \varepsilon,$$

where $v_g(x) := \int_\Omega g(d) e^{ikx.d}\, ds(d)$ is a Herglotz wave (see Lemma 3.1.2 in [9]). Now notice that the functions v_g and $\Phi(\,.\,,z)$ are the solutions of Helmholtz equation in G, hence

$$\left\| \Phi(\,.\,,z) - v_g \right\|_{H^1(G)} \leqslant c_1 \varepsilon.$$

Thus for every z and τ we can find function $g_\tau(z,\,.\,) \in L^2(\Omega)$, such that

$$\left\| \Phi(\,.\,,z) - v_{g_\tau} \right\|_{H^1(G)} \leqslant \tau.$$

If we consider the trace of $\Phi(\,.\,,z)$ and v_{g_τ} on ∂D, from Theorem 3.37 in [8], we see that

$$\left\| \Phi(\,.\,,z) - v_{g_\tau} \right\|_{H^{1/2}(\Gamma_D)} \leqslant c_2 \left\| \Phi(\,.\,,z) - v_{g_\tau} \right\|_{H^1(D)} \leqslant c_2 \tau,$$

where c_2 is a constant depends only on D. Also from Lemma 4.3 in [8], we have

$$\left\| \frac{\partial}{\partial \nu} \Phi(\,.\,,z) - \frac{\partial}{\partial \nu} v_{g_\tau} \right\|_{H^{-1/2}(\Gamma_I)} \leqslant c_3 \tau.$$

Therefore for every $\tau > 0$ and $z \in B \setminus \bar{D}$, there is $g_\tau(z,\,.\,) \in L^2(\Omega)$ such that

$$\left\| \Phi^s(\,.\,,z) - v_{g_\tau}^s \right\|_{H^1(B \setminus \bar{D})} \leqslant C\tau \tag{11}$$

and

$$\left\| \Phi^\infty(\,.\,,z) - v_{g_\tau}^\infty \right\|_{L^2(\Omega)} \leqslant C\tau, \tag{12}$$

where C depends only on D and B, while $v^s_{g_\tau}$ and $v^\infty_{g_\tau}$ are the scattered field and the far field with respect to the Herglotz wave v_{g_τ}.

Let $D_\rho = \{z \in \mathbb{R}^m \mid d(z, D) \leqslant \rho\}$. Then according to the regularity of the solution of elliptic equation and the relation (11) we conclude that for every $x \in B \setminus D_\rho$,

$$\left\| \Phi^s(\,.\,, z) - v^s_{g_\tau} \right\|_{H^2(B_\rho(x))} \leqslant C_1 \left\| \Phi^s(\,.\,, z) - v^s_{g_\tau} \right\|_{H^1(B_\rho(x))} \leqslant C_1 C\tau,$$

where B_ρ is a ball with center in x and radius ρ, moreover C_1 depends on ρ. Also the imbedding theorem and the above result imply that $\Phi^s(\,.\,, z) - v^s_{g_\tau}$ is a Hölder continuous function on B_ρ and we have,

$$\left| \Phi^s(x, z) - v^s_{g_\tau}(x) \right| \leqslant C_2 \left\| \Phi^s(\,.\,, z) - v^s_{g_\tau} \right\|_{H^2(B_\rho(x))} \leqslant C_2 C_1 C\tau = C_\rho \tau, \tag{13}$$

where C_ρ depends only on ρ, B and D.

On the other hand, we know that

$$v^s_{g_\tau}(x) = \int_\Omega g_\tau(z, d) u^s(x, d)\, ds(d), \tag{14}$$

for every $x \in \mathbb{R}^m \setminus \bar{D}$. Moreover

$$v^\infty_{g_\tau}(\hat{x}) = \int_\Omega g_\tau(z, d) u^\infty(\hat{x}, d)\, ds(d), \tag{15}$$

for every $\hat{x} \in \Omega$. Thus from (13), (14) and Theorem 5 we have

$$\left| \Phi^s(x, z) - \frac{1}{\gamma_m} \int_\Omega g_\tau(z, d) \Phi^\infty(-d, x)\, ds(d) \right| \leqslant C_\rho \tau.$$

Now from (12) and (15) we conclude that there is $g_\eta(x, \,.\,)$ such that

$$\left\| \Phi^\infty(\,.\,, x) - \int_\Omega g_\eta(x, \tilde{d}) u^\infty(\,.\,, \tilde{d})\, ds(\tilde{d}) \right\|_{L^2(\Omega)} \leqslant C\eta.$$

Thus

$$\left| \int_\Omega g_\tau(z, d) \left\{ \Phi^\infty(-d, x) - \int_\Omega g_\eta(x, \tilde{d}) u^\infty(-d, \tilde{d})\, ds(\tilde{d}) \right\} ds(d) \right|$$
$$\leqslant \left\| g_\tau(z, \,.\,) \right\|_{L^2(\Omega)} . \left\| \Phi^\infty(\,.\,, x) - v^\infty_{g_\eta} \right\|_{L^2(\Omega)} \leqslant C\eta \left\| g_\tau(z, \,.\,) \right\|_{L^2(\Omega)}.$$

Therefore for every $x, z \in B \setminus D_\rho$, we have

$$\left| \Phi^s(x, z) - \frac{1}{\gamma_m} \int_\Omega \int_\Omega g_\eta(x, \tilde{d}) g_\tau(z, d) u^\infty(-d, \tilde{d})\, ds(\tilde{d})\, ds(d) \right|$$
$$\leqslant C_\rho \tau + \frac{C\eta}{\gamma_m} \left\| g_\tau(z, \,.\,) \right\|_{L^2(\Omega)}.$$

Now we formulate the previous calculations and results in the following theorem. In order to do this we define the backprojection operator, Q as

$$(Qw)(x,z) := \frac{1}{\gamma_m}\int_\Omega\int_\Omega g_\eta(x,\tilde{d})g_\tau(z,d)u^\infty(-d,\tilde{d})\,ds(\tilde{d})\,ds(d).$$

Theorem 7. *Consider a mixed boundary condition obstacle D which is contained in a ball B. For every τ and η, there are kernels g_τ and g_η such that*

$$\left|\Phi^s(z,z) - (Qu^\infty)(z,z)\right| \leqslant C_\rho\tau + \frac{C\eta}{\gamma_m}\left\|g_\tau(z,\,.)\right\|_{L^2(\Omega)},$$

for every $z \in \mathbb{R}^m \setminus D_\rho$, moreover C and C_ρ are constants.

Remark 1. For an appropriate choice of τ and η the above error can be made arbitrarily small. In fact for given $\tau > 0$, we can choose η such that $\eta\|g_\tau(z,\,.)\|_{L^2(\Omega)}$ becomes sufficiently small. Now if $\tau \to 0$, and $\eta(\tau) \to 0$, then the error tends to zero.

We now summarize the singular sources method step by step.

(1) With a priori knowledge $D \subset B$, choose a domain approximation $G(z)$ for each $z \in B$ such that $z \notin G(z)$ and the unknown inclusion $D \subset G(z)$ is valid as far as possible.
(2) Choose value τ and then calculate the density $g_\tau(z,\,.)$.
(3) Choose η according to the above remark, then calculate $g_\eta(z,\,.)$.
(4) Calculate the backprojection $(Qu^\infty)(z,z)$ and determine the boundary as the set of points where $(Qu^\infty)(z,z)$ is large. In fact these points are located in the ρ-neighborhood of the boundary, Γ.

Remark 2. In order to apply this method, we need to choose the region $G(z)$ with the property $\bar{D} \subset G(z)$, but this seems impossible when D is unknown. There are some strategies to take care of this trouble. Here we mention one of them which is used in [9,10]. We start with a number of fixed directions $p_1,\dots,p_8$ which divided the plane in 8 symmetric region. For every direction p_i, we choose a special region $G_i(z)$ and compute $a_i^{(1)}(z)$ as an approximation $\Phi^s(z,z)$ using the operator Q, where Q is depending on $G_i(z)$. We can obtain a first approximation D_1 to the domain D as the set

$$D_1 := \left\{z \in B\colon \left|a_i^{(1)}(z)\right| > C \text{ for } i = 1,\dots,8\right\}.$$

In each further step, we adapt the choice $G(z)$ according to the reconstruction D_n of the nth step, $\bar{D}_n \subset G(z)$, and repeat the procedure to obtain the $(n+1)$th approximation D_{n+1}. For more detail, the reader is referred to [9,10].

Remark 3. We estimate $\Phi^s(z,z)$ from the far field, u^∞ by the operator Q, and if u_δ^∞ is measured as the far field u^∞, with some noise such that

$$\left\|u^\infty - u_\delta^\infty\right\|_{L^2(\Omega\times\Omega)} \leqslant \delta,$$

then the error for the approximation of $\Phi^s(z,z)$ by $(Qu_\delta^\infty)(z,z)$ is estimated by

$$\begin{aligned}\left|\Phi^s(z,z)-\left(Qu_\delta^\infty\right)(z,z)\right| &\leqslant \left|\Phi^s(z,z)-(Qu^\infty)(z,z)\right|+\left|Q\left(u^\infty-u_\delta^\infty\right)(z,z)\right|\\ &\leqslant C_\rho\tau+\frac{C\eta}{\gamma_m}\left\|g_\tau(z,\,.\,)\right\|_{L^2(\Omega)}+\frac{\delta}{\gamma_m}\left\|g_\tau(z,\,.\,)\right\|_{L^2(\Omega)}\left\|g_\eta(z,\,.\,)\right\|_{L^2(\Omega)}.\end{aligned}$$

Therefore the ill-posedness of the reconstruction of D is mainly influenced by the norm of the densities g_τ and g_η.

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